NORTH-HOLLAND

Block Pivoting and Shortcut Strategies for Detecting Copositivity

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ABSTRACT

In recently proposed quadratic optimization algorithms, copositivity detection procedures are frequently employed which deliver a feasible direction yielding a negative value of the considered quadratic form, if the answer is negative. To improve the computational performance of this routine, here (1) recursive characterizations of copositivity are presented which enable efficient reduction of the dimension of the problem using block pivoting techniques, and (2) shortcut strategies are described which are connected with diagonalization.

1. INTRODUCTION

In recently proposed global optimization algorithms [3, 4] for quadratic programming problems (QPs), the key procedure consists of a copositivity check of the following form: consider a given symmetric $n \times n$ matrix Q and a given polyhedral cone $\Gamma = \{x \in \mathbb{R}^n \colon Dx \geq 0\}$ (where D is an $m \times n$ matrix and $x \geq y$ means $x_j \geq y_j$ for all j). Then determine whether Q is Γ-copositive, i.e. whether

$$x'Qx \geq 0 \qquad \text{for all} \qquad x \in \Gamma \tag{1}$$

(a prime $'$ denotes transposition). If not, determine also a direction $v \in \Gamma$ with $v'Qv < 0$. This task amounts to the QP

$$x'Qx \to \min! \qquad \text{subject to} \quad Dx \geq 0, \quad x \in \mathbb{R}^n. \tag{2}$$

Hence solving the copositivity problem is NP-hard; see [16]. Even checking whether a given feasible point is a local solution of a QP is NP-hard [18]. On

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the other hand, despite the fears expressed in [16], the difference between checking local and global optimality in QPs is not as large as generally in mathematical programming. Indeed, local optimality of a Karush-Kuhn-Tucker point in a QP can be characterized by one copositivity condition, while global optimality of a Karush-Kuhn-Tucker point is equivalent to at most $m+1$ of such conditions; see [9] for the case of negative semidefinite Q and [2] for general Q. Returning to the problem of detecting copositivity, there are many methods which return a yes-no answer (e.g. [7, 8, 11, 15], some for the special case $D = I_n$), and which have similar worst-case behavior. However, as pointed out in [24], hard problems may have easy instances. The main aim of this paper is to contribute methods which are able systematically to exploit advantageous features of the data whenever they occur.

An algorithm for solving (2) should either (i) yield the information that the optimal objective value of (2) is zero and hence Q is Γ-copositive, or (ii) deliver a feasible direction $v \in \Gamma$ of unbounded recession, i.e. $v'Qv < 0$. To the author's knowledge, the only finite and exact algorithms which perform this task can be found, or derived from arguments, in [22, p. 28], [23, Remark 4.1], [24], and [3]. The last paper contains a recursive procedure which reduces the dimensionality of the problem, using a branch-and-bound method. This article deals with strategies to improve this approach from the viewpoint of computational cost.

This paper is structured as follows. After partitioning the problem in Section 2, we deal in Section 3 with a block pivoting method in order to reduce the height of the problem tree generated by the branch-and-bound method (for a similar approach to the related linear complementarity problem see [13]). Section 4 is devoted to diagonalization methods and related shortcuts, which in part are described in [26].

2. PARTITIONING THE COPOSITIVITY PROBLEM

Assume that the data and the variables of the problem, i.e. the matrices Q and D as well as the vector x, are partitioned in the following way, ignoring the ordering of coordinates:

$$Q = \begin{bmatrix} A & | & B \\ B' & | & C \end{bmatrix}, \qquad D = [E \quad | \quad F], \qquad \text{and} \qquad x = \begin{bmatrix} y \\ z \end{bmatrix}, \tag{3}$$

where A is a principal submatrix of order $k \times k$. Accordingly, B, C, E, and F are submatrices of order $k \times (n-k)$, $(n-k) \times (n-k)$, $m \times k$, and $m \times (n-k)$, respectively, while $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{n-k}$. Then the key relation is

$$x'Qx = y'Ay + 2y'Bz + z'Cz. \tag{4}$$

Thus Q is Γ-copositive if and only if for all fixed $z \in \mathbb{R}^{n-k}$, the following QP in the variable $y \in \mathbb{R}^k$,

$$g_z(y) = y'Ay + 2y'Bz + z'Cz \to \min! \qquad \text{subject to} \quad Ey \geq -Fz, \tag{5}$$

has a nonnegative optimal value. This means that for all $z \in \mathbb{R}^{n-k}$, the problem (5) has a global minimizer y_z with $g_z(y_z) \geq 0$, if (5) is feasible (recall that we as usual put $\inf \emptyset = +\infty$).

While the procedure in [1] and [3] emerges as a special case of the above arguments with $k = 1$ [and hence (5) is almost trivial to solve], the same approach has been followed in [24] with $k = n - 1$ or $k = n - 2$, rendering (5) as a parametric QP. However, since we regard the copositivity problem as an ordinary QP, we here reformulate the condition that (5) has a nonnegative optimal value again in terms of copositivity, arriving at a recursive criterion where in each step of recursion, the dimension of the problem will be reduced. Furthermore, we are not confined to small or large values of k, so that we may choose maximal dimension reduction (k large) at the cost of higher computational effort, or smaller problems (k small) at the cost of slower progress in the recursion, or a tradeoff between these possibilities.

In general, there may be two reasons why a QP has no solution: either it is infeasible, or the objective value is unbounded from below, in which case the feasible region clearly has to be unbounded. Thus to cope with directions of unbounded recession, we have to introduce the following cone:

$$\Gamma_0 = \{w \in \mathbb{R}^k \colon Ew \geq 0\}. \tag{6}$$

Next we consider a seemingly new copositivity notion, which generalizes the concept of conditional nonnegative-definite-plus in [23]:

DEFINITION 1. Let A be a symmetric $k \times k$ matrix and Γ, Δ be two (polyhedral) cones in $\mathbb{R}^k$. Then A is said to be Γ-*copositive-plus with respect to* Δ if

(i) A is Γ-copositive, and
(ii) $w'Aw = 0$ with $w \in \Gamma$ implies $w \in \Delta$.

REMARK 1. Note that this definition generalizes the following copositivity notions: (a) strict copositivity (put $\Delta = \{0\}$), (b) copositivity ($\Delta = \mathbb{R}^n$), and (c) copositive-plusness ($\Delta = \ker A = \{w \in \mathbb{R}^k \colon Aw = 0\}$).

Furthermore, the following implications hold for any choice of Δ and Γ:

$$\begin{aligned} A \text{ is strictly } \Gamma\text{-copositive} &\Rightarrow A \text{ is } \Gamma\text{-copositive-plus w.r.t. } \Delta \\ &\Rightarrow A \text{ is } \Gamma\text{-copositive.} \end{aligned}$$

Finally, observe that if $\Gamma \subseteq \ker A$, then A is Γ-copositive-plus w.r.t. Δ if and only if $\Gamma \subseteq \Delta$.

LEMMA 1. *The objective function of* (5) *is bounded from below for all* $z \in \mathbb{R}^{n-k}$ *if and only if* A *is* Γ_0*-copositive-plus with respect to the (polyhedral) cone*

$$\Delta = \{w \in \mathbb{R}^k\colon w'(Ay+Bz) \geq 0 \text{ if } Ey+Fz \geq 0,\ y \in \mathbb{R}^k,\ z \in \mathbb{R}^{n-k}\}, \tag{7}$$

which is the dual cone of the image $-[A \mid B](\Gamma)$.

Proof. First let us note that the feasible region of (5) is unbounded if and only if (5) is feasible and Γ_0 is not trivial. In this case, any y with $Ey \geq -Fz$ and any $w \in \Gamma_0 \setminus \{0\}$ will yield an unbounded ray $\{y+tw\colon t \geq 0\}$ of feasible points to (5). Looking at the leading terms in t of

$$g_z(y+tw) = t^2 w'Aw + 2tw'[Ay+Bz] + g_z(y), \quad t \geq 0, \tag{8}$$

the assertion follows immediately. ■

Next let us investigate the Karush-Kuhn-Tucker points of (5), i.e., the points satisfying the first-order optimality conditions. To this end, let e_i' and f_i' denote the ith rows of E and F, respectively (hence $e_i \in \mathbb{R}^k$ and $f_i \in \mathbb{R}^{n-k}$). For a subset $I \subseteq \{1, \ldots, m\}$, let E_I and F_I be the corresponding submatrices of E and F, with rows $\{e_i'\colon i \in I\}$ and $\{f_i'\colon i \in I\}$, respectively. The proof of the following result is similar to that of the fundamental theorem of linear programming; see, e.g. [14, pp. 19f.].

LEMMA 2. *Suppose that* $z \in \mathbb{R}^{n-k}$ *admits a Karush-Kuhn-Tucker point* y_z *of the problem* (5). *Then there is a subset* $I \subseteq \{1, \ldots, m\}$ *(possibly empty) having the following properties:*

(a) *the matrix* E_I *has full row rank;*
(b) *the set* I *consists of indices of constraints binding at* $\left[\begin{smallmatrix} y_z \\ z \end{smallmatrix}\right]$, *i.e.* $E_I y_z + F_I z = 0$, *or, equivalently,*

$$e_i' y_z + f_i' z = 0 \qquad \textit{for all} \quad i \in I; \tag{9}$$

(c) *if $I \neq \emptyset$, then associated with I there is a nonnegative Lagrange multiplier vector λ_I such that*

$$Ay_z + Bz = \tfrac{1}{2}E_I'\lambda_I; \tag{10}$$

if $I = \emptyset$, then $Ay_z + Bz = 0$.

Proof. The Karush-Kuhn-Tucker conditions can be formulated as follows. Let

$$L_z(y;\mu) = g_z(y) - \mu'[Ey + Fz]$$

be the corresponding Lagrange function. Then there exists a vector $\mu \in \mathbb{R}^m$ of Lagrange multipliers $\mu_i \geq 0$ for all i, not necessarily unique, such that

$$\nabla_y L_z(y_z;\mu) = 2(Ay_z + Bz) - E'\mu = 0 \qquad \text{(optimality condition)},$$

where $\nabla_y f = [\partial f/\partial y_j]_{1\leq j\leq k}$ denotes the gradient of a function f w.r.t. y; and

$$\mu_i = 0 \text{ or } e_i'y_z + f_i'z = 0,\ 1 \leq i \leq m \qquad \text{(complementarity condition)}. \tag{11}$$

Let $\widetilde{I} = \{i \in \{1,\dots,m\}\colon e_i'y_z + f_i'z = 0\}$ denote the set of all binding constraints. If $\widetilde{I} = \emptyset$, no constraints are binding at y_z and $\nabla g_z(y_z) = 0$, so that $I = \emptyset$ satisfies the above conditions in a trivial way. Otherwise, at least one constraint is binding at y_z. Now suppose that $E_{\widetilde{I}}$ has not full row rank, i.e., there are $\alpha_i \in \mathbb{R}$ such that $\sum_{i\in\widetilde{I}} \alpha_i e_i = 0$, and such that the set $I^+ = \{i \in \widetilde{I}\colon \alpha_i > 0\}$ is nonvoid. Choose an index $i_0 \in I^+$ with $\mu_{i_0}/\alpha_{i_0} \leq \mu_i/\alpha_i$ for all $i \in I^+$. Now put $I = \widetilde{I}\setminus\{i_0\}$ and

$$\lambda_i = \mu_i - \mu_{i_0}\frac{\alpha_i}{\alpha_{i_0}} \geq 0 \quad \text{for all } i \in I \quad \text{and} \quad \lambda_i = 0, \quad \text{otherwise.}$$

Then, by virtue of the optimality condition and (11),

$$\sum_{i\in I} \lambda_i e_i = \sum_{i\in\widetilde{I}} \mu_i e_i = 2(Ay_z + Bz),$$

so that also λ satisfies (10) and (11) instead of μ. Hence the λ_i are an alternative set of Lagrange multipliers. If E_I has full row rank, we are done. Otherwise repeat the preceding argument, replacing $\widetilde{I}$ with the smaller index set I. ■

In order to express y_z in terms of z, we follow an approach combining the active-set method with what is called the displacement method in structural analysis [20, 21]. This means that we first calculate the Lagrange multipliers λ and then obtain y_z. To this end we introduce the $m \times (n-k)$ matrix

$$H = EA^{-1}B - F, \tag{12}$$

provided A is nonsingular; put $J = \{1, \ldots, m\} \backslash I$, where I is as in Lemma 2; and partition E, F, and H accordingly:

$$E = \begin{bmatrix} E_I \\ E_J \end{bmatrix}, \qquad F = \begin{bmatrix} F_I \\ F_J \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} H_I \\ H_J \end{bmatrix} = \begin{bmatrix} E_I A^{-1} B - F_I \\ E_J A^{-1} B - F_J \end{bmatrix}.$$

Furthermore, let A_I denote the nonsingular matrix

$$A_I = E_I A^{-1} E_I'. \tag{13}$$

LEMMA 3. *Suppose that $z \in \mathbb{R}^{n-k}$ admits a Karush-Kuhn-Tucker point y_z of the problem* (5), *and that $I \subseteq \{1, \ldots, m\}$ is as in Lemma* 2. *Then:*

(a) *If A is nonsingular, y_z is of the form*

$$y_z = A^{-1}\big(E_I' A_I^{-1} H_I - B\big) z. \tag{14}$$

(b) *If $I \neq \emptyset$ and E_I is a square matrix, then y_z can be expressed as*

$$y_z = -E_I^{-1} F_I z. \tag{15}$$

Proof.

(a) Starting from (10), we obtain $y_z = \frac{1}{2} A^{-1} E_I' \lambda_I - A^{-1} B z$, where λ_I is as in Lemma 2. But then (9) yields

$$-F_I z = E_I y_z = \tfrac{1}{2} E_I A^{-1} E_I' \lambda_I - E_I A^{-1} B z = \tfrac{1}{2} A_I \lambda_I - E_I A^{-1} B z$$

and therefore

$$\lambda_I = 2 A_I^{-1} (E_I A^{-1} B - F_I) z = 2 A_I^{-1} H_I z, \tag{16}$$

so that we finally arrive at the assertion (14).

(b) Here we start directly from (9), which gives the desired result. ∎

It remains to establish a simple connection between z and the set I specified in Lemma 2. To this end, we have to introduce the following polyhedral cones which take into account both primal and dual feasibility:

$$\Gamma_I = \big\{ z \in \mathbb{R}^{n-k} \colon A_I^{-1} H_I z \geq 0 \text{ and } K_I z \geq 0 \big\} \tag{17}$$

with

$$K_I = E_J A^{-1} E_I' A_I^{-1} H_I - H_J$$

if A is nonsingular, and

$$\Gamma_I^{\square} = \{z \in \mathbb{R}^{n-k}\colon (E_I')^{-1}(B - AE_I^{-1}F_I)z \geq 0 \quad \text{and} \\ (F_J - E_J E_I^{-1} F_I)z \geq 0\} \tag{18}$$

if E_I is square and nonsingular.

LEMMA 4. *A point $z \in \mathbb{R}^{n-k}$ admits a Karush-Kuhn-Tucker point y_z of the problem* (5) *with a set $I \subseteq \{1, \ldots, m\}$ satisfying Lemma* 2,

(a) *if and only if $z \in \Gamma_I$, provided A is nonsingular;*
(b) *if and only if $z \in \Gamma_I^{\square}$, provided E_I is square.*

Proof.

(a) Necessity is obvious from (16), (14), and the primal feasibility condition $E_J y_z + F_J z \geq 0$. To show sufficiency, let

$$\lambda = \begin{bmatrix} 2A_I^{-1} H_I z \\ 0 \end{bmatrix},$$

and define y_z as in (14). Then, evidently, the point $\left[\begin{smallmatrix} y_z \\ z \end{smallmatrix}\right]$ and the multipliers λ_i satisfy the Karush-Kuhn-Tucker conditions.

(b) To establish necessity for this case, we have to calculate the Lagrange multipliers explicitly. From (10) we get $2(Ay_z + Bz) = E_I' \lambda_I$ and hence via Lemma 3(b)

$$0 \leq \lambda_I = 2(E_I')^{-1}(B - AE_I^{-1}F_I)z.$$

The second relation in the definition of $\Gamma_I^{\square}$ in (18) is again the primal feasibility condition. Sufficiency follows as in (a), by taking y_z from (15) and determining

$$\lambda = \begin{bmatrix} \lambda_I \\ 0 \end{bmatrix}$$

from (10). ∎

REMARK 2. Straightforward calculations show $\Gamma_I = \Gamma_I^{\square}$ if both A and E_I are nonsingular $k \times k$ matrices. Cf. Remark 6 below.

3. BLOCK PIVOTING IN COPOSITIVITY PROCEDURES

In one step of recursion, the algorithm in [3] reduces the dimension of the problem by one. Hence it creates a problem tree of height n and considerable width. It is the aim of this section to show that by block pivoting strategies based on the partition arguments of the preceding section, one can obtain a tree with significantly smaller height. Note that parts (a) of Theorems 5 and 6 below are the block variants of Theorems 5 and 6 in [1]. However, here we also incorporate part (b), the backtracking step in the case of a negative answer. This step must be performed recursively, to enrich a $z \in \Gamma_I \subseteq \mathbb{R}^{n-k}$ by a vector $y_z \in \mathbb{R}^k$ to obtain a direction $v \in \Gamma$. Hence assume that A is nonsingular and let—cf. (14)—

$$y_z(I) = A^{-1}E_I'A_I^{-1}H_I z - A^{-1}Bz \qquad \text{for all} \quad I \in \mathcal{I}, \tag{19}$$

where $\mathcal{I}$ denotes the system of the subsets I according to Lemma 2:

$$\begin{aligned}\mathcal{I} &= \{I \subseteq \{1, \ldots, m\}\colon E_I \text{ has full row rank}\} \\ &\subseteq \{I \subseteq \{1, \ldots, m\}\colon I \text{ has at most } k \text{ elements}\}.\end{aligned} \tag{20}$$

Finally, define the symmetric $(n-k) \times (n-k)$ matrices Q_I as follows:

$$Q_I = C - B'A^{-1}B + H_I'A_I^{-1}H_I. \tag{21}$$

Note that using the usual laws for empty matrices, we obtain for $I = \emptyset$

$$Q_\emptyset = C - B'A^{-1}B, \tag{22a}$$

$$\Gamma_\emptyset = \{z \in \mathbb{R}^{n-k}\colon (F - EA^{-1}B)z \geq 0\}, \tag{22b}$$

and

$$y_z(\emptyset) = -A^{-1}Bz. \tag{22c}$$

THEOREM 5. *Suppose that the symmetric $n \times n$ matrix Q is partitioned as in* (3) *with A a* nonsingular *principal submatrix of order k. Define Γ_0, Δ, $\mathcal{I}$, Γ_I, Q_I and $y_z(I)$ as in* (6), (7), (20), (17), (21), *and* (19), *respectively.*

(a) *Then Q is (strictly) Γ-copositive if and only if*
  (a1) *A is Γ_0-copositive-plus w.r.t. Δ (strictly Γ_0-copositive), and*
  (a2) *the $(n-k) \times (n-k)$ matrices Q_I are (strictly) Γ_I-copositive for all $I \in \mathcal{I}$.*
(b) *If one of the conditions* (a1), (a2) *does not hold, a direction $v \in \Gamma$ violating (strict) copositivity of Q can be obtained as follows:*

(b1) *If* $w \in \Gamma_0 \setminus \{0\}$ *satisfies* $w'Aw < 0$ ($w'Aw \leq 0$), *then*

$$v = \begin{bmatrix} w \\ 0 \end{bmatrix} \in \Gamma \setminus \{0\} \quad \textit{yields } v'Qv < 0 \ (v'Qv \leq 0).$$

If $w \in \Gamma_0$ *satisfies* $w'Aw = 0$ *but* $w'(Ay + Bz) < 0$ *for some* y, z *with* $Ey + Fz \geq 0$, *then*

$$v = \begin{bmatrix} y + tw \\ z \end{bmatrix} \in \Gamma \quad \textit{yields} \quad v'Qv < 0,$$

provided one chooses

$$t = \begin{cases} 1 & \textit{if } \ g_z(y) \leq 0, \\ -g_z(y)/w'(Ay + Bz) & \textit{if } \ g_z(y) > 0. \end{cases}$$

(b2) *If* $z \in \Gamma_I \setminus \{0\}$ *satisfies* $z'Q_I z < 0$ ($z'Q_I z \leq 0$) *for some* $I \in \mathcal{I}$, *then*

$$v = \begin{bmatrix} y_z(I) \\ z \end{bmatrix} \in \Gamma \setminus \{0\} \quad \textit{yields} \quad v'Qv < 0 \ \ (v'Qv \leq 0).$$

Proof. We proceed as follows. First we shall show sufficiency of (a1) and (a2) for the case of strict copositivity and for mere copositivity. Then (b) is proved, which also implies the necessity of (a1) and (a2). Let us deal with strict copositivity first: as is evident from (4), Q is strictly Γ-copositive if and only if (i) A is strictly Γ_0-copositive (this corresponds to $z = 0$ but $y \neq 0$) and (ii) for all $z \neq 0$ the QP (5) has a positive optimal objective value. So suppose that (5) is feasible. Then (a1) and Lemma 1 imply that there is a solution y_z of (5). Since all constraints in (5) are linear, no additional constraint qualifications are required to guarantee that y_z satisfies the Karush-Kuhn-Tucker conditions. Now Lemma 2 and Lemma 4(a) guarantee that $z \in \Gamma_I$ for some $I \in \mathcal{I}$. With λ_I as in Lemma 2 and

$$\lambda = \begin{bmatrix} \lambda_I \\ 0 \end{bmatrix},$$

equation (10) yields

$$Ay_z = \tfrac{1}{2}E'\lambda - Bz \quad \text{and} \quad y_z = \tfrac{1}{2}A^{-1}E'\lambda - A^{-1}Bz, \tag{23}$$

so that

$$\begin{aligned} g_z(y_z) - z'Cz &= \left[\tfrac{1}{2}A^{-1}E'\lambda - A^{-1}Bz\right]'\left[\tfrac{1}{2}E'\lambda - Bz\right] \\ &\quad + 2\left[\tfrac{1}{2}A^{-1}E'\lambda - A^{-1}Bz\right]'Bz \\ &= \left[\tfrac{1}{2}\lambda'EA^{-1} - z'B'A^{-1}\right]\left[\tfrac{1}{2}E'\lambda - Bz\right] \\ &\quad + 2\left[\tfrac{1}{2}\lambda'EA^{-1}Bz - z'B'A^{-1}Bz\right] \\ &= \tfrac{1}{4}\lambda'EA^{-1}E'\lambda - \lambda'EA^{-1}Bz + z'B'A^{-1}Bz \\ &\quad + \lambda'EA^{-1}Bz - 2z'B'A^{-1}Bz \\ &= \tfrac{1}{4}\lambda'EA^{-1}E'\lambda - z'B'A^{-1}Bz. \end{aligned} \tag{24}$$

Hence the optimal value is the sum of two quadratic forms in z and λ, respectively:

$$g_z(y_z) = z'(C - B'A^{-1}B)z + \tfrac{1}{4}\lambda'EA^{-1}E'\lambda. \tag{25}$$

From (14) follows the relation $E'\lambda = 2(Ay_z + Bz) = 2E_I'A_I^{-1}H_I z$, and hence the optimal objective value of (5) is given by

$$g_z(y_z) = z'(C - B'A^{-1}B)z + z'\left(E_I'A_I^{-1}H_I\right)'A^{-1}E_I'A_I^{-1}H_I z = z'Q_I z.$$

Thus the sufficiency part is established for strict copositivity.

Proceeding similarly for mere copositivity, we first observe Lemma 1 ensures that condition (a1) implies boundedness from below of the objective function in (5) for all z. Hence for any z generating a feasible QP (5), this problem has an optimal solution y_z, and $g_z(y_z) = z'Q_I z \geq 0$. Therefore sufficiency of both conditions follows also in this case.

The first assertion in (b1) is evident from $Dv = Ew \geq 0$, while the second follows from the relation (8), which yields $v'Qv = g_z(y + tw) < 0$ for the choice of $t > 0$ specified. To show the assertions in (b2), observe that by the definitions of $y_z(I)$ and Q_I, we have

$$v'Qv = g_z(y_z(I)) = z'Q_I z,$$

which also holds for $I = \emptyset$, in which case $\lambda = 0$; cf. (10), (11), and (22). Clearly, necessity of conditions (a1) and (a2) is implied by (b). ∎

REMARK 3. The expressions defining Γ_I and Q_I can be derived alternatively using block pivoting methods used for QPs (see, e.g., [12, 25]). Define $w = \lambda/2$, $v = Dx = Ey + Fz$; and introduce the slack variables $u = Ay + Bz - E'w$. Furthermore denote by $q = x'Qx - y'u - v'w =$

$(z'B')y - (z'F')w + z'Cz$. These relations can be arranged in the tableau

$$\begin{array}{r|ccc|c}
 & y & w_I & w_J & 1 \\
\hline
u = & A & -E_I' & -E_J' & Bz \\
v_I = & E_I & 0 & 0 & F_I z \\
v_J = & E_J & 0 & 0 & F_J z \\
\hline
q = & z'B' & -z'F_I' & -z'F_J' & z'Cz
\end{array}$$

Now primal, or dual, feasibility for the QP (5) amounts to $v \geq 0$, or $w \geq 0$, respectively, while the optimality condition (10) can be expressed as $u = 0$. Similarly, the complementarity property (11) amounts to $w'v = 0$, which specializes to $v_I = 0$ and $w_J = 0$ under the assumptions in Lemma 2, using an obvious notation. Exchanging first y with u and then v_I with w_I by the usual rules, we arrive at the following tableaux, where irrelevant entries are partly replaced with asterisks:

$$\begin{array}{r|ccc|c}
 & u & w_I & w_J & 1 \\
\hline
y = & A^{-1} & A^{-1}E_I' & * & -A^{-1}Bz \\
v_I = & E_I A^{-1} & A_I & * & -H_I z \\
v_J = & E_J A^{-1} & E_j A^{-1} E_I' & * & -H_J z \\
\hline
q = & z'B'A^{-1} & (H_I z)' & * & z'Q_{\emptyset} z
\end{array}$$

$$\longrightarrow \quad \begin{array}{r|ccc|c}
 & u & v_I & w_J & 1 \\
\hline
y = & * & * & * & y_z(I) \\
w_I = & * & * & * & A_I^{-1} H_I z \\
v_J = & * & * & * & K_I z \\
\hline
q = & * & * & * & z'Q_I z
\end{array}$$

Now the feasibility conditions $w_I \geq 0$ and $v_J \geq 0$ yield (17), while the condition $q = x'Qx \geq 0$ in the last, optimal tableau gives the copositivity of Q_I as defined in (21). Equation (19) is obtained similarly. Note that the last two tableaux coincide if $I = \emptyset$, the middle rows and columns being empty; cf. (22).

REMARK 4. Recursive application of the preceding theorem yields a finite procedure for detecting (strict) copositivity, provided property (a1) in Theorem 5 can be checked by a similar characterization. At the cost of slightly more notational effort, this certainly can be accomplished. But note that (a1) necessarily holds if either A is positive definite or $\Gamma_0 = \{0\}$. See Section 4 for a procedure to test for triviality of polyhedral cones.

REMARK 5. To obtain the matrices A^{-1} and A_I^{-1} occurring in the definitions of Γ_I and Q_I, one may employ Cholesky factorizations if A is positive definite and k is large. Furthermore, one can use the procedure in [5] to detect whether A is positive definite, and use the information provided in a shortcut if A is not positive definite; see Remark 9 below.

Since increasing the order k of the submatrix A might increase the cardinality of $\mathcal{I}$ and hence the number of generated subproblems in the branch-and-bound procedure, this recursive criterion might increase the width, but considerably reduces the height of the problem tree, compared to [1, Theorem 5(ii)]. Given k, one should of course choose that A among all nonsingular (or all positive definite: cf. Remark 4) principal $k \times k$ submatrices yielding a system $\mathcal{I}$ with the fewest elements, to keep also the width of the problem tree as small as possible. One may also incorporate a data-driven device to balance height and width in that a decision is taken whether to choose the maximal possible k (cf. Section 4) at the cost of large $\mathcal{I}$, or a smaller k resulting in fewer successor nodes of the tree.

Recursive dimensional reduction along the lines of Theorem 5 is, in principle, possible until Q is the zero matrix. However, if one wishes to use only positive definite A to avoid explicit checking of condition (a1), then one has to stop if all diagonal elements of Q are nonpositive. This is a special instance (order 1) of the case of a negative semidefinite A, which we shall treat in the following result. If A is negative semidefinite, the objective function in (5) is concave and therefore always admits an optimal solution at a vertex of the feasible set, if this is nonvoid. Thus consider the system

$$\begin{aligned} \mathcal{I}^\square &= \{I \subseteq \{1,\dots,k\}\colon E_I \text{ is square and nonsingular}\} \\ &= \{I \in \mathcal{I}\colon I \text{ has } k \text{ elements}\}, \end{aligned} \tag{26}$$

and define for any $I \in \mathcal{I}^\square$ the symmetric $k \times k$ matrix

$$Q_I^\square = C - (E_I^{-1}F_I)'B - B'E_I^{-1}F_I + F_I'(E_I')^{-1}AE_I^{-1}F_I. \tag{27}$$

THEOREM 6. *Suppose that the symmetric $n \times n$ matrix Q is partitioned as in* (3) *with A a* negative semidefinite *principal submatrix of order k. Define Γ_0, Δ, $\mathcal{I}^\square$, $\Gamma_I^\square$, and $Q_I^\square$ as in* (6), (7), (26), (18), *and* (27), *respectively.*

(a) *Then Q is (strictly) Γ-copositive if and only if*
 (a1) $\Gamma_0 \subseteq \ker A \cap \Delta$ $(\Gamma_0 = \{0\})$, *and*

(a2) *the* $(n-k)\times(n-k)$ *matrices* $Q_I^{\square}$ *are (strictly)* $\Gamma_I^{\square}$*-copositive for all* $I \in \mathcal{I}^{\square}$.

(b) *If one of the conditions* (a1), (a2) *does not hold, one obtains a direction* $v \in \Gamma$ *violating (strict) copositivity of* Q *as follows:*

(b1) *If* $w \in \Gamma_0$ *satisfies* $Aw \neq 0$ $(w \neq 0)$, *then*

$$v = \begin{bmatrix} w \\ 0 \end{bmatrix} \in \Gamma \setminus \{0\} \quad \textit{yields} \quad v'Qv < 0 \quad (v'Qv \leq 0).$$

If $w \in \Gamma_0$ *satisfies* $Aw = 0$ *but* $w'Bz < 0$ *for some* y, z *with* $Ey + Fz \geq 0$, *then*

$$v = \begin{bmatrix} y + tw \\ z \end{bmatrix} \in \Gamma \quad \textit{yields} \quad v'Qv < 0,$$

provided one chooses

$$t = \begin{cases} 1 & \textit{if} \;\; g_z(y) \leq 0, \\ -g_z(y)/w'Bz & \textit{if} \;\; g_z(y) > 0. \end{cases}$$

(b2) *If* $z \in \Gamma_I^{\square} \setminus \{0\}$ *satisfies* $z'Q_I^{\square}z < 0$ $(z'Q_I^{\square}z \leq 0)$ *for some* $I \in \mathcal{I}^{\square}$, *then*

$$v = \begin{bmatrix} -E_I^{-1}F_I z \\ z \end{bmatrix} \in \Gamma \setminus \{0\} \quad \textit{yields} \quad v'Qv < 0 \quad (v'Qv \leq 0).$$

Proof. Since one optimal solution y_z of the QP (5) now has to be a vertex of the feasible set $\{y\colon Ey \geq -Fz\}$, one can choose a nonsingular $k \times k$ submatrix E_I of E with $E_I y_z = -F_I z$. Indeed, if I denotes the set of binding constraints at y_z and the rank of E_I were less than k, then there would be a vector $v \neq 0$ with $E_I v = 0$ such that $y_z \pm v$ is also feasible for (5). But then $y_z = \frac{1}{2}(y_z + v) + \frac{1}{2}(y_z - v)$ could not be a vertex of the feasible set (cf. [14, p. 22]). Hence the rank of E_I must be k, and without loss of generality we can assume that E_I is square. Furthermore note that Γ_0-copositivity of a negative semidefinite matrix A is equivalent to the inclusion $\Gamma_0 \subseteq \ker A$. The remainder of the proof is similar to the argument of Theorem 5, but using Lemma 3(b) and Lemma 4(b), as well as Remark 1. ∎

REMARK 6. If A is negative definite, then obviously $Q_I = Q_I^{\square}$ and $y_z(I) = -E_I^{-1}F_I z$ if $I \in \mathcal{I}^{\square}$ and $z \in \Gamma_I = \Gamma_I^{\square}$. In this case condition (a1) in Theorem 6 amounts to the triviality of Γ_0 even if Q is merely Γ-copositive. However, for general nonsingular A the solution y_z of (5) need not be a vertex of the feasible set, and hence in this case we have to investigate $\mathcal{I}$ instead of the smaller system $\mathcal{I}^{\square}$.

If necessary, the relation in Theorem 6(a1) can be checked by establishing the dual inclusions $A(\mathbb{R}^k) \subseteq \Gamma_0^* = \{E'v\colon v \geq 0\}$ and $[A \mid B](\Gamma) \subseteq -\Gamma_0^*$, which are evident if both A and B are zero matrices, a case which may occur after diagonalization (cf. Section 4).

In the following theorem we give some shortcuts which can be used for instance in connection with Theorem 5 in the case $I = \emptyset$.

THEOREM 7. *Suppose that Q is partitioned as in Theorem* 5, *and that $G = C - B'A^{-1}B$ is negative semidefinite (e.g. because C is so and A is positive definite, as in* [3, (1.12)]). *Again, let $H = EA^{-1}B - F$. Then:*

(a) *$Q_\emptyset = G$ is $\Gamma_\emptyset$-copositive if and only if all columns g_i of G and their negative multiples $-g_i$ are representable in the form $\pm g_i = H'u_i^\pm$ for some $u_i^\pm \in \mathbb{R}^{n-k}$ satifying $u_i^\pm \geq 0$.*

(b) *If there is some i such that $\pm g_i$ is not representable as in* (a), *then any feasible point z_0 of the LP*

$$\pm g_i'z \to \max! \qquad \textit{subject to} \quad z \in \Gamma_\emptyset \tag{28}$$

with $\pm g_i'z_0 > 0$ yields a direction $v \in \Gamma$ such that $v'Qv < 0$ as follows:

$$y_0 = -A^{-1}Bz_0 \quad \textit{and} \quad v = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

Proof.

(a) Clearly $G = Q_\emptyset$ is $\Gamma_\emptyset$-copositive if and only if

$$Gz = 0 \qquad \text{whenever} \quad z \in \Gamma_\emptyset.$$

But this means that the kernel of G must contain $\Gamma_\emptyset$. Dualizing this inclusion yields

$$G(\mathbb{R}^{n-k}) = (\ker G)^* \subseteq \Gamma_\emptyset^* = H'(\mathbb{R}_+^{n-k}).$$

This relation in turn is satisfied if and only if $\pm g_i \in \Gamma_\emptyset^*$ holds for all $i \in \{1, \ldots, n-k\}$.

(b) If $\pm g_i'z_0 > 0$, then $Gz_0 \neq 0$ and hence $z_0'Gz_0 < 0$ because of negative semidefiniteness of G. But then also

$$v'Qv = g_{z_0}(y_{z_0}) = z_0'Gz_0 < 0$$

and $v \in \Gamma$ due to the definition of $y_0 = y_{z_0}(\emptyset)$; see (22c). ∎

REMARK 7. Whether $\pm g_i \in H'(\mathbb{R}_+^{n-k})$ can easily be checked, e.g. by using phase I of the simplex algorithm to obtain a point of the polyhedron $\{u \geq 0\colon H'u = \pm g_i\}$.

The remainder of this section is devoted to the classical case $D = I_n$, the $n \times n$ identity matrix, i.e. where $\Gamma = \mathbb{R}_+^n$ is the nonnegative orthant. Clearly the set I can belong to $\mathcal{I}$ only if $I \subseteq \{1, \ldots, k\}$. So the only I with k elements is $I = \{1, \ldots, k\}$. Then $E_I = I_k$, $F_I = 0$, $A_I = A^{-1}$, $H_I = -A^{-1}B$, $Q_I = C$, $\Gamma_I = \{z \in \mathbb{R}_+^{n-k}\colon Bz \geq 0\}$, and $y_z(I) = 0$. If I has $l < k$ elements, then up to coordinate permutations $E_I = [I_l \mid 0]$. Partitioning

$$A = \left[\begin{array}{c|c} S_I & R_{IJ} \\ R'_{IJ} & S_J \end{array}\right] \quad \text{and} \quad B = \begin{bmatrix} B_I \\ B_J \end{bmatrix}, \tag{29}$$

we shall show in the proof of Theorem 8 below that in the present case

$$Q_I = C - B'_J S_J^{-1} B_J, \tag{30}$$

$$y_z(I) = \left(A^{-1}E'_I A_I^{-1} H_I - A^{-1}B\right)z = \begin{bmatrix} 0 \\ -S_J^{-1}B_J \end{bmatrix} z = \begin{bmatrix} 0 \\ -S_J^{-1}B_J z \end{bmatrix}, \tag{31}$$

and

$$\Gamma_I = \left\{z \in \mathbb{R}_+^{n-k}\colon S_J^{-1}B_J z \leq 0 \quad \text{and} \quad \left(B_I - R_{IJ}S_J^{-1}B_J\right)z \geq 0\right\}. \tag{32}$$

THEOREM 8. *Suppose that Q is partitioned as in Theorem 5 with A positive definite. For $I \subseteq \{1, \ldots, k\}$, let Q_I, Γ_I, and $y_z(I)$ be as in (30), (32), and (31), respectively. Then Q is $\mathbb{R}_+^n$-copositive if and only if the $(n-k) \times (n-k)$ matrices*

$$Q_I \text{ are } \Gamma_I\text{-copositive} \qquad \text{for all } I \subseteq \{1, \ldots, k\}.$$

If $z'Q_I z < 0$ for some $I \subseteq \{1, \ldots, k\}$, then the direction

$$v = \begin{bmatrix} y_z(I) \\ z \end{bmatrix} \in \mathbb{R}_+^n \quad \textit{satisfies} \quad v'Qv < 0.$$

Proof. Now we use the technique sketched in Remark 3. Denote by $u = Ay + Bz$, perform a block pivot step at the center block to proceed from the upper to the lower tableau:

$$
\begin{array}{r|cc|c}
 & y_I & y_J & 1 \\
\hline
u_I = & S_I & R_{IJ} & B_I z \\
u_J = & R'_{IJ} & S_J & B_J z \\
\hline
q = & z'B'_I & z'B'_J & z'Cz
\end{array}
$$

$$
\longrightarrow \quad
\begin{array}{r|cc|c}
 & y_I & u_J & 1 \\
\hline
u_I = & * & R_{IJ}S_J^{-1} & (B_I - R_{IJ}S_J^{-1}B_J)z \\
y_J = & * & S_J^{-1} & -S_J^{-1}B_J z \\
\hline
q = & * & z'B'_J S_J^{-1} & z'(C - B'_J S_J^{-1} B_J)z
\end{array}
$$

Alternatively, we could have shown directly that the quantities defined in (30), (31), and (32) coincide with those defined in (21), (19), and (17), using Schur complements [6] to calculate A_I and H_I. Then Theorem 5 yields the desired result. ■

REMARK 8. If $B = 0$, then obviously $\Gamma_I = \mathbb{R}^{n-k}_+$. On the other hand, if $B_I - R_{IJ}S_J^{-1}B_J \neq 0$, then the successor cones Γ_I may be proper subsets of the nonnegative orthant $\mathbb{R}^{n-k}_+$. Hence Theorem 8 is not a mere paraphrase of [19, Theorem 3.14] or [22, Theorem 5.2], but rather has connections with [22, Theorem 4.1]. Note furthermore that the counterpart of Theorem 6 yields the following, quite straightforward characterization if A is *negative semidefinite*: Q is $\mathbb{R}^n_+$-copositive if and only if (i) $A = 0$; (ii) $b_{ij} \geq 0$ for all i, j; and (iii) C is $\mathbb{R}^{n-k}_+$-copositive.

EXAMPLE 1. Consider Example 4.1 from [22], where

$$
Q = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & -3 & -3 \\ 1 & -3 & 5 & 6 \\ 2 & -3 & 6 & 5 \end{bmatrix} \qquad \text{and} \qquad D = I_4.
$$

We choose as A the leading 2×2 positive definite principal submatrix, so that

$$
A^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
$$

and $I \subseteq \{1, 2\}$ for all $I \in \mathcal{I}$. Hence we have to treat four subproblems, two of which are trivial: indeed, we have $\Gamma_{\{1,2\}} = \Gamma_{\{2\}} = \{0\}$. But also the remaining cases are easy to deal with, because all successor cones are contained in the nonnegative orthant, and because the corresponding matrices

$$Q_\emptyset = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad Q_{\{1\}} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

have nonnegative entries only. Hence all matrices Q_I are Γ_I-copositive, and therefore also Q is $\mathbb{R}^4_+$-copositive. Note that in [22] ten tableaux have to be calculated to establish this property.

4. DIAGONALIZATION AND RELATED SHORTCUTS

Let us first treat the question how to choose from Q an appropriate positive definite principal submatrix A. To this end define

$$k_+(Q) = \text{ the number of positive eigenvalues of } Q. \tag{33}$$

If $k_+(Q) = n$, the copositivity problem is trivial. However, if $k_+(Q) < n$, it may be impossible to choose a positive definite principal submatrix A, as the example

$$Q = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = -3$ shows. On the other hand, $k_+(Q)$ is an upper bound for the order k of a positive definite principal submatrix A. Similar bounds apply to the case of a negative semidefinite principal submatrix A.

LEMMA 9. *Let A be a $k \times k$ principal submatrix of a symmetric $n \times n$ matrix Q, and let $k_+(Q)$ be as in* (33). *If A is a positive definite, then $k \leq k_+(Q)$. If A is negative semidefinite, then $k \leq n - k_+(Q)$.*

Proof. See, e.g., [25, Theorem 4.5]. ∎

Hence a method to guarantee the block partition of Q as required in the preceding section is to use a diagonalization of Q at first. So suppose we are provided with a diagonal $n \times n$ matrix T and a nonsingular $n \times n$ matrix U such that

$$T = U'QU. \tag{34}$$

Usual inertia arguments yield that T has $k_+(Q)$ positive entries.

REMARK 9. If the spectral decomposition of Q is known exactly beforehand, U may consist of the normalized eigenvectors of Q, with T containing the eigenvalues of Q. But, in general, spectral decomposition is not a finite, exact procedure. Instead, one may employ, e.g., Lagrange's reduction (see e.g. [17], pp. 236–245]) or any other finite, exact diagonalization procedure. For instance, one may proceed as in [5] to perform the first steps in Fig. 1 below. Similar approaches for QPs can also be found in [10].

Now replace Q with T, and also replace D with the corresponding transform DU, so that now $DU = [E \mid F]$ where E and F are matrices of order $m \times k$ and $m \times (n-k)$, respectively, while $B = 0$ and A and C are diagonal matrices. Furthermore, if one wants a maximal reduction of dimension in the recursive criterion one can always choose $k = k_+(Q)$ here, so that the diagonal elements of A are positive, while those of C are nonpositive. Hence C is negative semidefinite, implying that the assumption of Theorem 7 is satisfied in the case $k = k_+(Q)$. So the shortcut for $I = \emptyset$ can be performed.

Next let us turn to shortcuts using the diagonalization of Q. Denote the columns of U by $u_1, \ldots, u_n$, and assume without loss of generality that the diagonal entries τ_i of T satisfy $\tau_i < 0$ for all $i \in \{1, \ldots, r\}$ but $\tau_i \geq 0$ if $i > r$. Let us assume for the rest of the paper that $r \geq 1$ holds (otherwise Q is positive semidefinite and therefore copositive for all cones Γ). To reduce computational effort avoiding unnecessary recursion steps, we may:

(a) Check whether $u_i \in \Gamma \cup -\Gamma$ for some $i \in \{1, \ldots, r\}$. In this case we obtain $v = \pm u_i \in \Gamma$ with $v'Qv = \tau_i < 0$, and are done.

(b) Check whether Γ is contained in the linear span

$$L_+ = \mathrm{sp}(u_{r+1}, \ldots, u_n). \tag{35}$$

If the answer is affirmative, Γ-copositivity of Q is guaranteed, and we are done. Since in most applications Γ is given by a matrix D rather than by its (possibly numerous) extremal rays, we may dualize the inclusion we aim at, obtaining

$$L_+^\perp = \{x \in \mathbb{R}^n; u_i'x = 0 \text{ if } r < i \leq n\} \subseteq \Gamma^* = \{D'v \colon v \leq 0\}. \tag{36}$$

This relation can be checked using methods similar to that exhibited in Theorem 7, after obtaining the general solution of the homogeneous system $u_i'x = 0,\ r < i \leq n$. Note that (36) holds if and only if $L_+^\perp \subseteq \Gamma^\perp$, so that this shortcut can be ignored if we know that Γ has nonempty interior.

In [26], a threshold value for $k_+(Q)$ determines which of the shortcuts (a) or (b) is performed first, to save effort in a data-driven way. Furthermore, one may use the procedure Trivial (Γ) described in [3], which determines whether a given polyhedral cone Γ is trivial, generating a direction $v \in \Gamma \setminus \{0\}$ in the non-trivial case. This procedure essentially corresponds to phase I of the simplex algorithm, equipped with a dimension-reducing mechanism if Γ is contained in a proper linear subspace of $\mathbb{R}^n$. The latter feature can be dropped when calling Trivial($\Gamma \cap L_+^\perp$) in a shortcut refining step (a) above, which in the nontrivial case again yields a direction $v \in \Gamma$ with $v'Qv < 0$. Note that in case of negative definiteness, only a nontrivial Γ allows for a negative answer. Using these considerations, one may proceed as in Fig. 1 (taken from [26]), where diagonalization is obtained from the spectral decomposition [note that $L_+^\perp = L_- = \mathrm{sp}(u_1, \ldots, u_r)$ in this case].

Finally observe that if $B = 0$, then for $I \in \mathcal{I}$ the cone Γ_I as well as the matrix Q_I and the vector $y_z(I)$ simplify as follows:

$$\Gamma_I = \left\{z \in \mathbb{R}^{n-k}\colon A_I^{-1} F_I z \le 0 \ \text{ and } \ F_J z \ge E_J A^{-1} E_I' A_I^{-1} F_I z\right\}, \tag{37}$$
$$Q_I = C + F_I' A_I^{-1} F_I \qquad \text{and} \qquad y_z(I) = -A^{-1} E_I' A_I^{-1} F_I z. \tag{38}$$

Similar simplifications apply to $\Gamma_I^\square$ and $Q_I^\square$ if $I \in \mathcal{I}^\square$.

EXAMPLE 2. Consider

$$Q = \begin{bmatrix} 1 & 7 & 1 & 3 \\ 7 & 1 & 3 & 1 \\ 1 & 3 & 1 & 7 \\ 3 & 1 & 7 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Since D has no negative entries, Γ contains the nonnegative orthant $\mathbb{R}_+^4$, but also e.g. $v = [1, 1, -1, -1]'$, yielding $v'Qv = 4$. Proceeding without diagonalization, we can only choose A to be a 1×1 positive definite principal submatrix, say the leading one (observe that there are no negative semidefinite principal submatrices, and that Γ_0 is nontrivial for all partitions of Q). Then $E = [1, 1, 0, 0]'$, so that $I \subseteq \{1, 2\}$ for all $I \in \mathcal{I}$.

Starting with $I = \emptyset$, we calculate

$$F - EA^{-1}B = \begin{bmatrix} -7 & -1 & -3 \\ -7 & 0 & -3 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$Q_\emptyset = C - B'A^{-1}B = \begin{bmatrix} -48 & -4 & -20 \\ -4 & 0 & 4 \\ -20 & 4 & -8 \end{bmatrix},$$

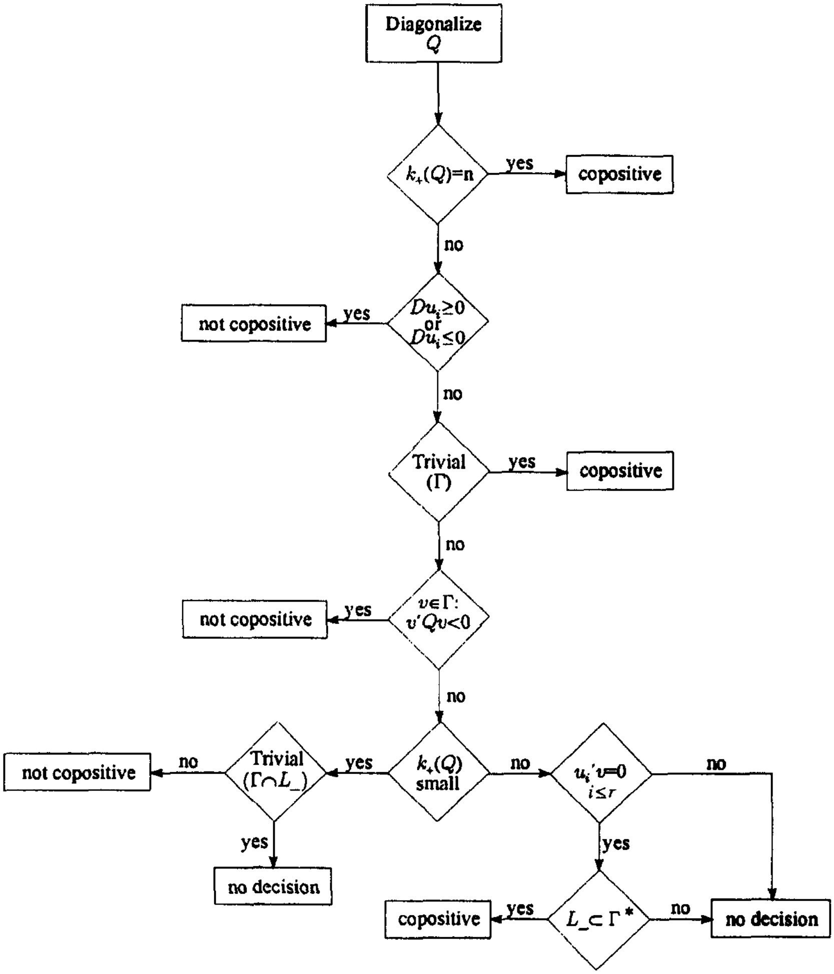

FIG. 1. Shortcuts using diagonalization.

so that $\Gamma_\emptyset = \{z \in \mathbb{R}^3\colon z_1 = z_3 = 0,\ z_2 \leq 0\}$ and therefore $Q_\emptyset$ is $\Gamma_\emptyset$-copositive.

For $I = \{1\}$, the matrix $E_I = 1$ is square and nonsingular, whence $A_I = 1$ results. Further, $F_I = 0'$ yields $H_I = [7, 1, 3]$ and also [cf. (27) and

(18) as well as Remarks 2 and 6]

$$F_J - E_J E_I^{-1} F_I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$Q_{\{1\}} = C - 0 = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 7 \\ 1 & 7 & 1 \end{bmatrix},$$

so that $\Gamma_{\{1\}} = \{z \in \mathbb{R}^3\colon z_1 \geq 0,\ z_2 \geq 0,\ z_1 + z_3 \geq 0\}$. A point of this cone which does not belong to $\mathbb{R}^3_+$ is given by $z = [1, 1, -1]'$ with $z'Q_{\{1\}}z = -7$. Since $-E_I^{-1}F_I = 0$, we obtain $v = [0, 1, 1, -1]' \in \Gamma$ with $v'Qv < 0$. For lucidity, we treat the same example via diagonalization. In this case, an exact spectral decomposition of Q is easy:

$$U = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{bmatrix}$$

with the corresponding eigenvalues $\lambda_1 = 4$, $\lambda_2 = 12$, $\lambda_3 = -4$, $\lambda_4 = -8$.

First observe that none of the above shortcuts succeeds. Indeed, $\Gamma \cap L_- = \{0\}$ and L_- is not a subset of Γ^*, since $u_3 = [-1, 1, 1, -1]'$ has no representation $D'w$ with $w \geq 0$; this is evident when looking at the last coordinate of w. Now we have to replace D with $\widetilde{D} = DU$ and hence E and F with $\widetilde{E} = EU$ and $\widetilde{F} = FU$, respectively. Rescaling, we choose $\widetilde{A}$ as the diagonal 2×2 matrix with 1 and 3 on the diagonal, while of course $\widetilde{C}$ has -1 and -2 as diagonal elements. Rescaling also D, we put

$$\widetilde{E} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \widetilde{F} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}.$$

Then evidently $\Gamma_\emptyset = \{0\}$, so that we can start by investigating $I = \{1\}$. Short calculations yield

$$\widetilde{F}_J - \widetilde{E}_J(\widetilde{E}_I)'(\widetilde{A}_I)^{-1}\widetilde{F}_I = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 1 & -5 \end{bmatrix}$$

and

$$\widetilde{Q}_{\{1\}} = \widetilde{C} + (\widetilde{F}_I)'(\widetilde{A}_I)^{-1}\widetilde{F}_I = \frac{1}{4}\begin{bmatrix} -1 & -3 \\ -3 & -5 \end{bmatrix},$$

so that after removing the redundant inequalities we have $\widetilde{\Gamma}_{\{1\}} = \{z \in \mathbb{R}^2\colon z_1 + 3z_2 \geq 0,\ z_1 - 5z_2 \geq 0\}$, with an extremal ray $\widetilde{z} = [5,1]'$ satisfying $\widetilde{z}'\widetilde{Q}_{\{1\}}\widetilde{z} < 0$. Calculating $y_{\widetilde{z}}(\{1\})$ via (38), we obtain $\widetilde{v} = [-3,-1,5,1]'$ and finally $v = U\widetilde{v} = [0,2,1,-2]' \in \Gamma$ with $v'Qv = -15 < 0$.

EXAMPLE 3. To illustrate the use of Theorem 6 and to contrast the method proposed here with that used in [3], which already improved that in [15], consider the data from Example 1 in [3] :

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & -1 & 2 \\ 0 & -5 & 4 \\ 5 & 0 & 3 \\ 0 & 5 & 4 \\ -2 & 1 & 2 \end{bmatrix}.$$

Choosing as A the leading negative definite principal submatrix of order 2, we first have to check the triviality of $\Gamma_0 = \{z \in \mathbb{R}^2\colon Ez \geq 0\}$, which is obvious. Next we have to investigate all $I \subset \{1,\ldots,5\}$ with $k = 2$ elements, with the exception of $I = \{2,4\}$, since $E_{\{2,4\}}$ is singular. Now we use (18), and find out the triviality of the cones $\Gamma_I^{\square}$ for all

$$I \in \{\{1,3\},\{1,4\},\{2,5\},\{3,5\}\}.$$

Indeed, in all of these cases we get nonzero $(E_I')^{-1}AE_I^{-1}F_I \leq 0$, so that $\Gamma_I^{\square} \subseteq \mathbb{R}_+$. Furthermore the first entry of $M_I = F_J - E_J'E_I^{-1}F_I$ is -8 or $-\frac{8}{5}$ if $I = \{1,3\}$ or $I = \{2,5\}$, respectively, while the last entry of M_I is $-\frac{1}{5}$ or -8 for $I = \{1,4\}$ or $I = \{3,5\}$, respectively. Hence these cones are trivial. On the other hand, (38) yields (cf. Remark 6) $Q_I^{\square} = Q_I = 1 - (E_I^{-1}F_I)'(E_I^{-1}F_I) = 0$ if

$$I \in \{\{1,2\},\{1,5\},\{2,3\},\{3,4\},\{4,5\}\},$$

so that $\Gamma_I^{\square}$-copositivity of $Q_I^{\square}$ results for all $I \in \mathcal{I}^{\square}$. Hence Q is Γ-copositive. Compared with the problem tree generated in Example 1 of [3], we had to deal with considerably fewer subproblems here, because the dimension of the problem has been reduced by two in one recursive step (see Fig. 2). However, Q is not strictly Γ-copositive since, e.g. for $I = \{1,2\}$, we have $Q_I^{\square} = 0$ but $\Gamma_I^{\square}$ is not trivial [since, as above, $(E_I')^{-1}AE_I^{-1}F_I \leq 0$,

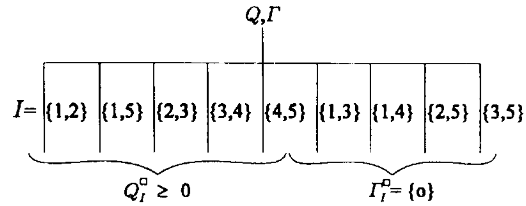

FIG. 2. Problem tree generated by block pivoting.

but here $M_I \geq 0$]. Indeed, for instance, $z = 5 \in \Gamma_I^\square$, and via (38) we obtain a direction $v = [3, 4, 5]' \in \Gamma \setminus \{0\}$ with $v'Qv = 0$.

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